Bergen County Mathematics League

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Brief Contest Solutions #2 **2010-2011**

2-1) The second and third digits can be 0, 2, 4, 6, or 8; but the first digit cannot be a 0, so the number of 3-digit numbers with all digits even is $4 \times 5 \times 5 = \boxed{100}$.

2-2) Let $g(n) = f(n) - f(n-2)$. Then, $g(4003) = f(4003) - f(4001) = 3f(4002)+1-[3f(4000)+1] = 3[f(4002)-f(4000)]$; so $g(4003) = 3 \cdot g(4002)$. Thus, g is an exponential function with general solution $g(n) = k \cdot 3^n$. Since $f(1)=0$, $f(2)=1$, and $f(3)=4$, $g(3) = k \cdot 3^3 = f(3)-f(1) = 4$, so $k = 4/3^3$ and $g(n) = k \cdot 3^n = 4 \cdot 3^{n-3}$. Finally, $f(4003)-f(4001) = g(4003) = \boxed{4 \times 3^{4000}}$.
[Note: $f(n) = \frac{1}{2}(3^{n-1}-1)$ is the general solution for f.]

2-3) When 4 is the altitude, the base-radius is $\frac{3}{2\pi}$ and volume is $4\pi\left(\frac{3}{2\pi}\right)^2 = \frac{9}{\pi}$. Similarly, when 3 is the altitude and $\frac{4}{2\pi} = \frac{2}{\pi}$ is the altitude, the volume is $3\pi\left(\frac{2}{\pi}\right)^2 = \frac{12}{\pi}$. The ratio is $\left(\frac{9}{\pi}\right) \div \left(\frac{12}{\pi}\right) = \boxed{\frac{3}{4}}$.

2-4) $x \div (y \div z) = (x)(z \div y) = \frac{xz}{y}$, and $(z \div x) \div y = \left(\frac{z}{x}\right)\left(\frac{1}{y}\right) = \frac{z}{xy}$, so the original expression $= \frac{xz}{y} \div \frac{z}{xy} = \frac{xz}{y} \times \frac{xy}{z} = \boxed{x^2}$.

2-5) Setting the numerator equal to 0, $x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x-1)(x^2-1) = 0$, so $x = 1$ or -1. Since $x = 1$ makes the denominator 0, the solution is $\boxed{-1}$.

2-6) $$\frac{2-\sqrt{8}+\sqrt{12}}{1+\sqrt{2}-\sqrt{3}} = \frac{2-2\sqrt{2}+2\sqrt{3}}{1+\sqrt{2}-\sqrt{3}} = \frac{2[(1-\sqrt{2})+(\sqrt{3})]}{(1+\sqrt{2})-(\sqrt{3})} \cdot \frac{(1+\sqrt{2})+\sqrt{3}}{(1+\sqrt{2})+\sqrt{3}} =$$
$$\frac{2(1+\sqrt{2}-\sqrt{2}-2+\sqrt{3}-\sqrt{6}+\sqrt{3}+\sqrt{6}+3)}{(1+2\sqrt{2}+2)-3} = \frac{\cancel{2}(2+2\sqrt{3})}{\cancel{2}\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}(\sqrt{2}+\sqrt{6})}{\cancel{2}} = \boxed{\sqrt{2}+\sqrt{6}}.$$