Bergen County Mathematics League

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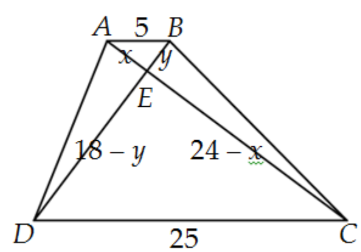
4-1. We want the thousands digit of N to be as large as possible. If the thousands digit is 9, then the leftmost digits of N must be 960, which contradicts the requirement that all digits are nonzero. If the thousands digit is 8, then the leftmost digits of N must be 848, which contradicts the requirement that all digits are distinct. If the thousands digit is 7, then the only possibility is that $N = \boxed{7248}$.

4-2. Let $2011p + 9 = n^2$, from which $2011p = n^2 - 9 = (n + 3)(n - 3)$. Since 2011 and p are both primes, one of the factors at the right must be 2011 and the other must be p. If $n + 3 = 2011$, then $n = 2008$, from which $n - 3 = 2005$. Since 2005 is divisible by 5, it is not a prime. Therefore, $n - 3 = 2011$ and $n = 2014$, from which $n + 3 = \boxed{2017}$.

4-3. Adding the equations, we get $x^2 + 2xy + y^2 = 121$, from which $(x + y)^2 = 121$, $x + y = 11$ or -11. If $x + y = 11$, then from $x^2 + xy = x(x + y) = x(11) = 77$, we get $x = 7$ and $y = 4$. If $x - y = -11$, then from $x^2 + xy = x(x + y) = x(-11)$, $= 77$ we get $x = -7$ and $y = -4$. Therefore, all possible ordered pairs (x, y) are $\boxed{(7, 4), (-7, -4)}$.

4-4. As shown in the diagram, $AB = 5$, $CD = 25$, $BD = 18$, and $AC = 24$. Let $AE = x$ and $BE = y$, we get $EC = 24 - x$ and $ED = 18 - y$. Since $\triangle ABE \sim \triangle CDE$, $\frac{AE}{CE} = \frac{BE}{DE} = \frac{AB}{CD}$, from which $\frac{x}{24-x} = \frac{y}{18-y} = \frac{5}{25}$. Thus $x = 4$ and $y = 3$. Clearly, $\triangle ABE$ and $\triangle CDE$ are right triangles, so the diagonals of $ABCD$ are perpendicular. The area of $ABCD$ is $\frac{1}{2}(18)(24) = \boxed{216}$.



4-5. Let $N = (2^m)(3^n)(k^6)$ for generality, from which $2N = (2^{m+1})(3^n)(k^6)$ is a perfect square and $3N = (2^m)(3^{n+1})(k^6)$ is a perfect cube. Clearly, $m + 1$ must be even and m must be a multiple of 3, and n must be even and $n + 1$ must be a multiple of 3. The smallest possible N occurs when $(m, n, k) = (3, 2, 1)$, so $N = (2^3)(3^2)(1^6) = \boxed{72}$.

4-6. Converting everything to base 3, we get $\log_6 27 = \frac{\log_3 27}{\log_3 6} = \frac{3}{\log_3 6} = \frac{3}{\log_3 3 + \log_3 2} = \frac{3}{1 + \log_3 2} = a$, from which $3 = a + a \log_3 2$. Thus $\log_3 2 = \frac{3-a}{a}$. Therefore, $\log_{18} 16 = \frac{\log_3 16}{\log_3 18} = \frac{\log_3 2^4}{\log_3 2 * 3^2} = \frac{4 \log_3 2}{\log_3 2 + \log_3 3^2} = \frac{4 \log_3 2}{\log_3 2 + 2} = \frac{4\left(\frac{3-a}{a}\right)}{\frac{3-a}{a} + 2} = \boxed{\frac{4(3-a)}{3+a}}$.