Bergen County Math League

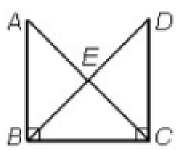
Contest #4 **2021** **Answers/Solutions**

4-1. **Answer:** 80

Method I: There are 8 such numbers from 12 to 19, 9 from 21 to 29, and 31 to 39, …, and 91 to 99. The total is $8+(9+9+9+9+9+9+9+9)=8+72=80$.

Method II: Excluding 0, $9\times 9=81$ numbers are possible. Disallowing the 11, leaves 80 numbers.

4-2. **Answer:** 18



Method I: Since $m\angle EAB=m\angle EBA=m\angle EBC=m\angle ECB=m\angle ECD=m\angle EDC=45°$, and $\overline{AB}\cong\overline{BC}\cong\overline{DC}$, it follows that $\Delta ABE\cong\Delta BCE\cong\Delta CDE$. Now, since $AC=12, AE=EC=6=EB$, and the are of ΔBCE is $\frac{1}{2}(6)(6)=18$.

Method II: Draw $\overline{AD}$, creating square $ABCD$ whose diagonal is 12. The area of any square is half the product of its diagonals, so the area of $ABCD$ is 72. But ΔBCE is only $\frac{1}{4}$ of the square, so its area is 18.

4-3. **Answer:** 5

Since $2^{2020}\times 5^{2021}=2^{2020}\times 5^{2020}\times 5^1=10^{2020}\times 5=50\ldots 0$, the sum of all the digits in the product is 5.

4-4. **Answer:** 15

When x is divided by y, the quotient is 3 and the remainder is 7, so $x=3y+7$. Divide this by $2y$ to get $\frac{x}{2y}=\frac{3y}{2y}+\frac{7}{2y}=\frac{2y}{2y}+\frac{y}{2y}+\frac{7}{2y}=1+\frac{y+7}{2y}$. When we divide x by y, the remainder is 7, so $y>7$. Since y is integral, $y\geq 8$, making $0<\frac{y+7}{2y}<1$. Hence, the quotient, q, is 1 and the remainder, r, is $y+7$. Since $y\geq 8$, the least possible value of r is 15.

4-5. **Answer:** 128

Method I: In this progression, with first term a and common difference d, the sum of the third and fifth terms is $a+2d+a+4d=14$. Simplifying, $a+3d=7$. Next, $129=a+(a+d)+\cdots+(a+11d)=12a+66d=12a+22(3d)=12a+22(7-a)=-10a+154$, so $a=2\frac{1}{2}$ and $d=1\frac{1}{2}$. Finally, if $193=a+(n-1)d$, then it follows that $n=128$.

Method II: Since $a+3d$ is the fourth term, and since the value of this term is 7 (see Method I), the first 12 terms are $7-3d, 7-2d, 7-d, 7, 7+d, \ldots, 7+8d$. Their sum is $84+30d=129$, so $d=1\frac{1}{2}$. The solution now follows the last sentence of Method I.

4-6. **Answer:** $\frac{4}{7}$

Method I: Ali wins whenever one of the sequences of $H, TTTH, TTTTTTH, \ldots$ occurs: sequences where the first H is preceded by $3n$ T's, whose probability is $\left(\frac{1}{2}\right) + \left(\frac{1}{8}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{8}\right)^2\left(\frac{1}{2}\right) + \cdots$ This is an infinite geometric series whose sum is $\frac{4}{7}$.

Method II: Let Ali have probability p of winning. If Ali flips a tail on the first round, Bobby then finds himself having probability p of winning, for it's as though he's the first player in the game. Since the probability that Ali will flip a tail on round 1 is $\frac{1}{2}$, Bobby's actual probability of winning is $\frac{1}{2}p$. By the same reasoning, Carmen's probability of winning is half that of Bobby's, so Carmen's probability of winning is $\frac{1}{4}p$. Since $p + \frac{1}{2}p + \frac{1}{4}p = 1$, $p = \frac{4}{7}$.

Method III: The possible round one flips are $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. Ali wins in the first 4 cases and loses in the next 3. If all three people get tails, the procedure is repeated, with the same set of possible outcomes. Since there are only 7 ways in which the game concludes (the 7 ways listed), Ali's probability of winning is $\frac{4}{7}$.