Bergen County Math League

Contest #3 **2021-2022** **Answers/Solutions**

3-1. **Answer:** 32
If two positive numbers have a fixed product, their sum will be minimized when the numbers are as close together as possible. Thus, the smallest perimeter of such a rectangle is $4x5 = 20$, the perimeter of a square with side length 5. The largest such perimeter is that of a $1x25$ rectangle. Its perimeter is 52. The difference between the perimeters is $52 - 20 = 32$.

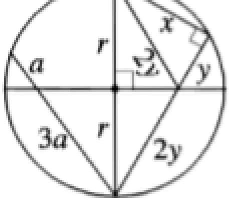
3-2. **Answer:** $1, 3, 5, 8, 8$
Since 8 is the only mode and 5 is the median, there must be two 8's and one 5 among the 5 integers. The mean of the 5 integers is 5, so their sum is $5x5 = 25$. Since $8 + 8 + 5 = 21$, the two missing numbers must total 4 and average 2. Since 8 is the only mode, there cannot be two 2's. Since all numbers are positive integers, the five numbers must be 1, 3, 5, 8, 8.

3-3. **Answer:** $\frac{1}{3}$
Intuitively, since tan is an increasing function in the first quadrant, this product will be maximized whenever $\tan x = \tan y$, *i.e.*, when $x = y = 30°$. At this point, $(\tan x)(\tan y) = \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3}$.
PROOF: Let's verify that the maximum possible product is actually $\frac{1}{3}$. Let $t = \tan x$; so $0 < t < \sqrt{3}$. Is $(\tan x)(\tan y) = (\tan x)(\tan(60° - x)) = t\left(\frac{\sqrt{3}-t}{1+t\sqrt{3}}\right) \leq \frac{1}{3}$ a valid inequality? Since *t* is positive, we can clear fractions without concern. Clearing fractions, we get $3t\sqrt{3} - 3t^2 \leq 1 + t\sqrt{3}$. That's true precisely when $t^2 - \frac{2t}{\sqrt{3}} + \frac{1}{3}$ is non-negative. This last expression is a perfect square – it's $\left(t - \frac{1}{\sqrt{3}}\right)^2$ – so the expression is always non-negative. Reversing these steps proves that the maximum value guessed is correct.

3-4. **Answer:** 27
Flash and Ace tie if Ace runs 75m while Flash runs 60m, so Flash runs at $\frac{4}{5}$ of Ace's speed. Continuing, Speedy runs at $\frac{4}{5}$ of Flash's speed, or $\frac{4}{5}\left(\frac{4}{5}\right) = \frac{16}{25} = \frac{48}{75}$ of Ace's speed. In meters, the head start needed by Speedy is $75 - 48 = 27$.

3-5. **Answer:** 4042
If $f(x) = ax + b$ and $g(x) = cx + d$, then $g\big(f(x)\big) = c(ax + b) + d = acx + (d + bc) = 2x + 6$. Thus $ac = 2$. Also, $f\big(g(x)\big) = a(cx + d) + b = acx + (b + ad)$. Since its graph passes through the origin, the constant term, $b + ad$, must equal 0. Therefore, $f\big(g(x)\big) = 2x$ and $f\big(g(2021)\big) = 2(2021) = 4042$.

3-6. **Answer:** $1\frac{1}{8}$



By the Pythagorean Theorem, $x^2 + (3y)^2 = (2r)^2$. Expanding, $x^2 + 9y^2 = 4r^2$. Furthermore, $x^2 + y^2 = (2y)^2$, or $x^2 = 3y^2$. By substitution, $12y^2 = 4r^2$, so $y = \frac{r}{\sqrt{3}}$. Thus $3y = r\sqrt{3} =$ the length of the longer chord. Similarly, for the shorter length:

$4r^2 - 16a^2 = 8a^2$. Thus, $4r^2 = 24a^2$ and $a = \frac{4}{\sqrt{6}}$ with the length of the shorter cord being $\frac{4r}{\sqrt{6}}$. Hence, the ratio of their chord-lengths, longer to shorter, is $\frac{3\sqrt{2}}{4}$. Its square is $1\frac{1}{8}$.