Bergen County Math League

Contest #5 **2021-2022** **Answers/Solutions**

5-1. **Answer:** 9

Since $23 = 9 + 9 + 4 + 1$, the largest square in this sum is 9.

[By Lagrange's 4-Square Theorem, every positive integer can be written as the sum of the squares of 4 integers. This is the only way to write 23 as such a sum.]

5-2. **Answer:** 256

Method I: Square each side of $n = \sqrt{x\sqrt{x\sqrt{x}}}$ three times to get $n^8 = x^7$. If this has a solution in integers, n must be the 7th power of some integer and x must be the 8th power of an integer. The smallest such $x > 1$ must be the 8th power of 2, so $x = 2^8 = 256$.

Method II: The innermost x will have its square root taken 3 times in succession, so the least $x > 1$ for which $\sqrt{x\sqrt{x\sqrt{x}}}$ is an integer must be a power of 2 for which the exponent can be divided by 2 three times in succession. That means that the exponent is $2^3 = 8$, so $x = 2^8 = 256$.

5-3. **Answer:** 30

We shall show that all ten numbers are equal. If not, then look at the largest one, which is the average of its two nearest neighbors. If one neighbor were smaller, the other would have to be larger. This is impossible, so all three are equal. Similarly, all ten numbers must be equal, so each of the ten numbers is 30.

5-4. **Answer:** (1,2)

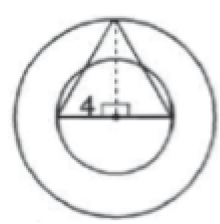
Clearly, $|x| \neq 1$. To find solutions, consider 2 cases:

Case I: If $|x| > 1$, the exponent must be negative. Since $x^2 - x - 2 = (x-2)(x+1) < 0$ $\Leftrightarrow -1 < x < 2$, the solutions in the interval $|x| > 1$ are $\{x|\ 1 < x < 2\}$.

Case II: If $|x| < 1$, the exponent must be positive. Since $x^2 - x - 2 = (x-2)(x+1) > 0$ $\Leftrightarrow x < -1\ or\ x > 2$, there are no solutions in the interval $|x| < 1$.

Only **Case I** works, so the solutions are $1 < x < 2$.

5-5. **Answer:** 48



The area of the smaller circle is 16π, so its radius is 4 and a side of the triangle is 8. The larger circle's radius, an altitude of the triangle, is $4\sqrt{3}$. The larger circle's area is $\pi(4\sqrt{3})^2 = 48\pi$. So we have $k = 48$.

5-6. **Answer:** 91

Call the integers a, b and 2015. To minimize $a + b$, make $a < b < 2015$. Since the longest side of a triangle < the sum of the other sides, $\frac{1}{a} < \frac{1}{b} + \frac{1}{2015}$, or $\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab} < \frac{1}{2015}$. Since $b - a \geq 1$, we get $\frac{1}{ab} \leq \frac{b-a}{ab} < \frac{1}{2015}$, so $ab > 2015$. Since $b > a$, we get $b^2 > 2015$, so $b \geq 45$. If $b =$

45, then $a \le 44$, so $ab \le 1980 < 2015$, so this doesn't work. If $b = 46$, then $a \le 45$. If $a \le 44$, then $\frac{1}{a} - \frac{1}{b} \ge \frac{1}{44} - \frac{1}{46} = \frac{1}{1012} > \frac{1}{2015}$. If $a = 45$, then $\frac{1}{a} - \frac{1}{b} = \frac{1}{2070} < \frac{1}{2015}$. This works, and $a + b = 91$. If $a + b < 91$ and $b \ge 47$, then $a \le 43$. But then we have $\frac{1}{a} - \frac{1}{b} \ge \frac{4}{2021} > \frac{1}{2015}$, which doesn't work.